An Asymptotic Formula for Goldbach's Conjecture with Monic Polynomials in $\mathbb{Z}[x]$

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1 Introduction.

In a recent MONTHLY note, Saidak [6], improving on a result of Hayes [1], gave Chebyshev-type estimates for the number $\mathfrak{R}(y) = \mathfrak{R}_f(y)$ of representations of the monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree $d > 1$ as a sum of two irreducible monics $g(x)$ and $h(x) \in \mathbb{Z}[x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in absolute value by y.

Here, we do not distinguish the sum $g(x)+h(x)$ from $h(x)+g(x)$, and whenever we write that a monic polynomial $p(x)$ in $\mathbb{Z}[x]$ is "irreducible", we mean irreducible over $\mathbb{Q}$. We observe that Saidak's argument with slight modifications gives that, for y sufficiently large,

$$c_1 y^{d-1} < \mathfrak{R}(y) < c_2 y^{d-1},$$

where c_1 and c_2 are constants that depend on the degree and the coefficients of the polynomial $f(x)$. In this note, we give a proof that the number $\mathfrak{R}(y)$ is asymptotic to $(2y)^{d-1}$, i.e.,

$$\lim_{y\to\infty} \frac{\mathfrak{R}(y)}{(2y)^{d-1}} = 1.$$

In fact, our approach implies that there is a constant c_3 depending only on d such that if y is sufficiently large, then

$$\mathfrak{R}(y) = (2y)^{d-1} + E, \qquad \text{where } |E| \le c_3 y^{d-2} \log y.$$

2 Preliminaries.

For functions $r(y)$ and $s(y)$, we write $r(y) = \mathcal{O}(s(y))$ if there is a constant $C > 0$ such that $|r(y)| \le Cs(y)$ for all sufficiently large y. If the constant C depends on a value d or on the coefficients and degree of a polynomial $f(x)$, we use instead $\mathcal{O}_d$ or $\mathcal{O}_f$, respectively.

First, we state the following lemma which implies that the probability that a monic polynomial in $\mathbb{Z}[x]$ of given degree whose second coefficient is fixed is reducible is 0 (that is, the density of reducible polynomials with bounded coefficients approaches 0 as the bound on the coefficients goes to infinity).

Lemma 1. *Let $d > 1$ be an integer and fix an integer g_{d-1}. For each $y \geq 2$, let r_y denote the number of d-tuples of integers $(g_{d-1}, g_{d-2}, \ldots, g_1, g_0)$ satisfying $-y \leq g_i \leq y$ for $i \in \{0, 1, \ldots, d-1\}$ such that the polynomial*

$$x^d + g_{d-1}x^{d-1} + \cdots + g_1 x + g_0$$

is reducible. (So, in particular, $r_y = 0$ if $y < |g_{d-1}|$.) Then $r_y = \mathcal{O}_d(y^{d-2} \log y)$.

In order to prove Lemma 1, we modify an argument of Pólya and Szegö [5, Pt. VIII, Ch. 5, no. 266], and we use an inequality that is a known consequence of Landau [2] and simple properties of Mahler measure [3, 4].

The Mahler measure, $M(p)$, of a polynomial $p(x) = \sum_{j=0}^{k} p_j x^j$ in $\mathbb{Z}[x]$ is

$$M(p) = \exp\left(\int_0^1 \ln |p(e^{2\pi i t})| \; dt\right).$$

Mahler showed that for $0 \leq j \leq k$, $|p_j| \leq \binom{k}{j} M(p)$ and remarked that $M(p)$ is multiplicative. Landau showed that $1 \leq M(p) \leq (p_k^2 + p_{k-1}^2 + \cdots + p_1^2 + p_0^2)^{\frac{1}{2}}$. From these, we deduce the following inequality which we state as our second lemma.

Lemma 2. *Let $g(x)$ be a polynomial in $\mathbb{Z}[x]$ of degree d of the form*

$$g(x) = g_d x^d + g_{d-1}x^{d-1} + \cdots + g_1 x + g_0$$

such that $g(x)$=$a(x)b(x)$, where $a(x)$ and $b(x)$ are in $\mathbb{Z}[x]$. Let $a(x)$ take the form

$$a(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0.$$

Then for $0 \leq l \leq m$, a_l satisfies

$$|a_l| \leq \binom{m}{l} (g_d^2 + g_{d-1}^2 + \cdots + g_1^2 + g_0^2)^{\frac{1}{2}}.$$

Proof of Lemma 1. We remind the reader that if $g(x) \in \mathbb{Z}[x]$ factors in $\mathbb{Q}[x]$ as a product of two polynomials of degree at least 1, then $g(x)$ factors in $\mathbb{Z}[x]$ as a product of two polynomials of degree at least 1. Now, let $g(x) \in \mathbb{Z}[x]$ be a reducible, monic polynomial of degree $d > 1$ such that all of its coefficients are in absolute value $\leq y$ and g_{d-1} is fixed as in the lemma. Then there exist two monic polynomials $a(x)$ and $b(x) \in \mathbb{Z}[x]$ of degree ≥ 1 such that $g(x) = a(x)b(x)$. Let us further take

$$\deg(a) = m \geq n = \deg(b),$$

where $m+n=d$. Note that there are at most $\lfloor \frac{d}{2} \rfloor$ possibilities for the pair (m,n). We write $a(x)$ and $b(x)$ in the following forms:

$$a(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$
$$b(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0.$$

Since the number of monic polynomials we are considering with $g_0 = 0$ is $\mathcal{O}_d(y^{d-2})$, it is sufficient to show that the number of d-tuples

$$(a_{m-1}, a_{m-2}, \ldots, a_1, a_0, b_{n-1}, b_{n-2}, \ldots, b_1, b_0)$$

as above with $a_0 b_0 \neq 0$ is equal to $\mathcal{O}_d(y^{d-2}\log y)$.

We consider $a(x)$ which has degree $m \leq d-1$. A similar argument applies to $b(x)$. For $1 \leq l \leq m-1$, Lemma 2 implies

$$|a_l| \leq \binom{m}{l}(1^2 + g_{d-1}^2 + \cdots + g_1^2 + g_0^2)^{\frac{1}{2}} \leq \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor}((d+1)y^2)^{\frac{1}{2}} = C_d y,$$

where C_d depends only on d. Thus, the number of $(d-4)$-tuples

$$(a_{m-2}, \ldots, a_1, b_{n-2}, \ldots, b_1)$$

is $\mathcal{O}_d(y^{m-2}y^{n-2}) = \mathcal{O}_d(y^{d-4})$.

Observe that when we multiply $a(x)$ by $b(x)$, since they are both monic polynomials, the value the coefficient g_{d-1} takes will result from the sum $a_{m-1} + b_{n-1}$. Also recall that g_{d-1} is fixed, so determining a_{m-1} also determines b_{n-1}. Hence the number of 2-tuples (a_{m-1}, b_{n-1}) is $\mathcal{O}(y)$.

Since $a_0 b_0 = g_0$, we have $1 \leq |a_0 b_0| \leq y$. Thus, the number of 2-tuples (a_0, b_0) is bounded by

$$4\sum_{q \leq y}\sum_{\delta | q} 1 = 4\sum_{\delta \leq y}\sum_{\substack{q \leq y \\ \delta | q}} 1 \leq 4\sum_{\delta \leq y}\frac{y}{\delta} = \mathcal{O}(y \log y),$$

where the 4 appears above since each of a_0 and b_0 may be either positive or negative. Combining this estimate with the above, the lemma follows. □

Remark. If we remove the condition in Lemma 1 that g_{d-1} is fixed, then $r_y = \mathcal{O}_d(y^{d-1}\log y)$. This is a direct consequence of a more general theorem of van der Waerden [7].

3 Theorem.

Theorem 1. *Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ of degree $d > 1$. The number $\mathfrak{R}(y)$ of representations of $f(x)$ as a sum of two irreducible monics $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in absolute value by y, is asymptotic to $(2y)^{d-1}$.*

Proof. Let $f(x)$ be a given monic polynomial in $\mathbb{Z}[x]$ of degree $d > 1$ that takes the form

$$x^d + f_{d-1}x^{d-1} + \cdots + f_1 x + f_0.$$

We are looking for pairs of monic polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$ with coefficients bounded in absolute value by y such that $f(x) = g(x) + h(x)$. Without loss of generality, let $\deg(g) > \deg(h)$, and observe that $\deg(g) = d$ and $1 \leq \deg(h) \leq d - 1$.

If $y \geq 1 + \max\{|f_0|, |f_1|, \ldots, |f_{d-1}|\}$, then the total number of pairs of monic (not necessarily irreducible) polynomials $g(x), h(x)$ is

$$\sum_{T=0}^{d-2} \prod_{t=0}^{T} (2\lfloor y \rfloor + 1 - |f_t|) = (2y)^{d-1} + \mathcal{O}_f(y^{d-2}) \sim (2y)^{d-1}.$$

We claim that almost all of these pairs of monic polynomials $g(x)$, $h(x)$ consist of two irreducible polynomials. Thus, $\mathfrak{R}(y) \sim (2y)^{d-1}$. We in fact establish

$$\mathfrak{R}(y) = (2y)^{d-1} + \mathcal{O}_d(y^{d-2} \log y)$$

by showing that there are $\mathcal{O}_d(y^{d-2} \log y)$ pairs of monic polynomials $(g(x), h(x))$ where at least one of $g(x)$ or $h(x)$ is reducible. Once a particular $g(x)$ or $h(x)$ is fixed, it determines the other. We count the ways $g(x)$ might be reducible separately from the ways $h(x)$ might be reducible.

First, we count the ways $g(x)$ might be reducible. We have that $\deg(g) = d$. Since h is monic and $\deg h \leq d - 1$, the equation $f(x) = g(x) + h(x)$ implies that either $g_{d-1} = f_{d-1}$ or $g_{d-1} = f_{d-1} - 1$, so in each of these cases, g_{d-1} is fixed. Since the coefficients of $g(x)$ are bounded in absolute value by y, we have met all the conditions for Lemma 1. Therefore we have that there are at most $\mathcal{O}_d(y^{d-2} \log y)$ monic reducible polynomials $g(x)$.

Next, we count the ways that the monic polynomial $h(x)$ might be reducible. Suppose we have that $\deg(h) = d-1$. Since the coefficients of $h(x)$ are bounded in absolute value by y, by our remark at the end of the previous section, we have at most $\mathcal{O}_d(y^{d-2} \log y)$ monic reducible polynomials $h(x)$ of degree $d-1 \geq 1$. If $\deg(h) < d - 1$, we note that our remark also implies that the number of ways $h(x)$ can be reducible is $\mathcal{O}_d(y^{d-2} \log y)$.

Thus,

$$\begin{aligned}
\mathfrak{R}(y) &= \left(\sum_{T=0}^{d-2} \prod_{t=0}^{T} (2\lfloor y \rfloor + 1 - |f_t|) \right) + \mathcal{O}_d(y^{d-2} \log y) + \mathcal{O}_d(y^{d-2} \log y) \\
&= \left((2y)^{d-1} + \mathcal{O}_f(y^{d-2}) \right) + \mathcal{O}_d(y^{d-2} \log y) \\
&= (2y)^{d-1} + \mathcal{O}_d(y^{d-2} \log y),
\end{aligned}$$

where we have used that any constant depending only on the coefficients and degree of $f(x)$ is small compared to $\log y$ when y is sufficiently large. The result follows. □

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