An Asymptotic Formula for Goldbach’s Conjecture with Monic Polynomials in $\mathbb{Z}[x]$

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1 Introduction.

In a recent Monthly note, Saidak [6], improving on a result of Hayes [1], gave Chebyshev-type estimates for the number $\mathcal{R}(y) = \mathcal{R}_f(y)$ of representations of the monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree $d > 1$ as a sum of two irreducible monics $g(x)$ and $h(x) \in \mathbb{Z}[x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in absolute value by $y$.

Here, we do not distinguish the sum $g(x) + h(x)$ from $h(x) + g(x)$, and whenever we write that a monic polynomial $p(x)$ in $\mathbb{Z}[x]$ is “irreducible”, we mean irreducible over $\mathbb{Q}$. We observe that Saidak’s argument with slight modifications gives that, for $y$ sufficiently large,

$$c_1 y^{d-1} < \mathcal{R}(y) < c_2 y^{d-1},$$

where $c_1$ and $c_2$ are constants that depend on the degree and the coefficients of the polynomial $f(x)$. In this note, we give a proof that the number $\mathcal{R}(y)$ is asymptotic to $(2y)^{d-1}$, i.e.,

$$\lim_{y \to \infty} \frac{\mathcal{R}(y)}{(2y)^{d-1}} = 1.$$ 

In fact, our approach implies that there is a constant $c_3$ depending only on $d$ such that if $y$ is sufficiently large, then

$$\mathcal{R}(y) = (2y)^{d-1} + E, \quad \text{where } |E| \leq c_3 y^{d-2} \log y.$$ 

2 Preliminaries.

For functions $r(y)$ and $s(y)$, we write $r(y) = O(s(y))$ if there is a constant $C > 0$ such that $|r(y)| \leq C s(y)$ for all sufficiently large $y$. If the constant $C$ depends on a value $d$ or on the coefficients and degree of a polynomial $f(x)$, we use instead $O_d$ or $O_f$, respectively.
First, we state the following lemma which implies that the probability that a monic polynomial in \( \mathbb{Z}[x] \) of given degree whose second coefficient is fixed is reducible is 0 (that is, the density of reducible polynomials with bounded coefficients approaches 0 as the bound on the coefficients goes to infinity).

**Lemma 1.** Let \( d > 1 \) be an integer and fix an integer \( g_{d-1} \). For each \( y \geq 2 \), let \( r_y \) denote the number of \( d \)-tuples of integers \( (g_{d-1}, g_{d-2}, \ldots, g_1, g_0) \) satisfying \(-y \leq g_i \leq y \) for \( i \in \{0, 1, \ldots, d - 1\} \) such that the polynomial

\[
x^d + g_{d-1}x^{d-1} + \cdots + g_1x + g_0
\]

is reducible. (So, in particular, \( r_y = 0 \) if \( y < |g_{d-1}| \).) Then \( r_y = \mathcal{O}(y^{d-2} \log y) \).

In order to prove Lemma 1, we modify an argument of Pólya and Szegö [5, Pt. VIII, Ch. 5, no. 266], and we use an inequality that is a known consequence of Landau [2] and simple properties of Mahler measure [3, 4].

The Mahler measure, \( M(p) \), of a polynomial \( p(x) = \sum_{j=0}^{k} p_j x^j \) in \( \mathbb{Z}[x] \) is

\[
M(p) = \exp \left( \int_0^1 \ln|p(e^{2\pi it})| \, dt \right).
\]

Mahler showed that for \( 0 \leq j \leq k \), \( |p_j| \leq \binom{k}{j} M(p) \) and remarked that \( M(p) \) is multiplicative. Landau showed that \( 1 \leq M(p) \leq (p_k^2 + p_{k-1}^2 + \cdots + p_1^2 + p_0^2)^{\frac{1}{2}} \).

From these, we deduce the following inequality which we state as our second lemma.

**Lemma 2.** Let \( g(x) \) be a polynomial in \( \mathbb{Z}[x] \) of degree \( d \) of the form

\[
g(x) = g_k x^d + g_{d-1} x^{d-1} + \cdots + g_1 x + g_0
\]

such that \( g(x) = a(x)b(x) \), where \( a(x) \) and \( b(x) \) are in \( \mathbb{Z}[x] \). Let \( a(x) \) take the form

\[
a(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0.
\]

Then for \( 0 \leq l \leq m \), \( a_l \) satisfies

\[
|a_l| \leq \binom{m}{l} (g_k^2 + g_{d-1}^2 + \cdots + g_1^2 + g_0^2)^{\frac{1}{2}}.
\]

**Proof of Lemma 1.** We remind the reader that if \( g(x) \in \mathbb{Z}[x] \) factors in \( \mathbb{Q}[x] \) as a product of two polynomials of degree at least 1, then \( g(x) \) factors in \( \mathbb{Z}[x] \) as a product of two polynomials of degree at least 1. Now, let \( g(x) \in \mathbb{Z}[x] \) be a reducible, monic polynomial of degree \( d > 1 \) such that all of its coefficients are in absolute value \( \leq y \) and \( g_{d-1} \) is fixed as in the lemma. Then there exist two monic polynomials \( a(x) \) and \( b(x) \in \mathbb{Z}[x] \) of degree \( \geq 1 \) such that \( g(x) = a(x)b(x) \).

Let us further take

\[
\deg(a) = m \geq n = \deg(b),
\]

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where \( m + n = d \). Note that there are at most \( \lfloor \frac{d}{2} \rfloor \) possibilities for the pair \((m,n)\). We write \( a(x) \) and \( b(x) \) in the following forms:

\[
\begin{align*}
a(x) &= x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \\
b(x) &= x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0.
\end{align*}
\]

Since the number of monic polynomials we are considering with \( g_0 = 0 \) is \( O_d(y^{d-2}) \), it is sufficient to show that the number of \( d \)-tuples

\[(a_{m-1}, a_{m-2}, \ldots, a_1, a_0, b_{n-1}, b_{n-2}, \ldots, b_1, b_0)\]

as above with \( a_0b_0 \neq 0 \) is equal to \( O_d(y^{d-2}\log y) \).

We consider \( a(x) \) which has degree \( m \leq d - 1 \). A similar argument applies to \( b(x) \). For \( 1 \leq l \leq m - 1 \), Lemma 2 implies

\[
|a_l| \leq \binom{m}{l}(1^2 + g_{d-1}^2 + \cdots + g_l^2 + g_0^2)^\frac{1}{2} \leq \left( \frac{d - 1}{\binom{d-1}{2}} \right)((d + 1)y^2)^\frac{1}{2} = C_dy,
\]

where \( C_d \) depends only on \( d \). Thus, the number of \( (d - 4) \)-tuples

\[(a_{m-2}, \ldots, a_1, b_{n-2}, \ldots, b_1)\]

is \( O_d(y^{m-2}y^{n-2}) = O_d(y^{d-4}) \).

Observe that when we multiply \( a(x) \) by \( b(x) \), since they are both monic polynomials, the value the coefficient \( g_{d-1} \) takes will result from the sum \( a_{m-1} + b_{n-1} \). Also recall that \( g_{d-1} \) is fixed, so determining \( a_{m-1} \) also determines \( b_{n-1} \). Hence the number of 2-tuples \((a_{m-1}, b_{n-1})\) is \( O(y) \).

Since \( a_0b_0 = g_0 \), we have \( 1 \leq |a_0b_0| \leq y \). Thus, the number of 2-tuples \((a_0, b_0)\) is bounded by

\[
4 \sum_{q \leq y} \sum_{\delta | q} 1 = 4 \sum_{d \leq y} \sum_{\delta | d} 1 \leq 4 \sum_{d \leq y} \frac{y}{d} = O(y \log y),
\]

where the 4 appears above since each of \( a_0 \) and \( b_0 \) may be either positive or negative. Combining this estimate with the above, the lemma follows. \( \Box \)

**Remark.** If we remove the condition in Lemma 1 that \( g_{d-1} \) is fixed, then \( r_y = O_d(y^{d-1}\log y) \). This is a direct consequence of a more general theorem of van der Waerden [7].

### 3 Theorem.

**Theorem 1.** Let \( f(x) \) be a monic polynomial in \( \mathbb{Z}[x] \) of degree \( d > 1 \). The number \( R(y) \) of representations of \( f(x) \) as a sum of two irreducible monics \( g(x) \) and \( h(x) \) in \( \mathbb{Z}[x] \), with the coefficients of \( g(x) \) and \( h(x) \) bounded in absolute value by \( y \), is asymptotic to \((2y)^{d-1}\). 

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Proof. Let \( f(x) \) be a given monic polynomial in \( \mathbb{Z}[x] \) of degree \( d > 1 \) that takes the form

\[
x^d + f_{d-1}x^{d-1} + \cdots + f_1x + f_0.
\]

We are looking for pairs of monic polynomials \( g(x) \) and \( h(x) \) in \( \mathbb{Z}[x] \) with coefficients bounded in absolute value by \( y \) such that \( f(x) = g(x) + h(x) \). Without loss of generality, let \( \deg(g) = \deg(h) \), and observe that \( \deg(g) = d \) and \( 1 \leq \deg(h) \leq d - 1 \).

If \( y \geq 1 + \max\{|f_0|, |f_1|, \ldots, |f_{d-1}|\} \), then the total number of pairs of monic (not necessarily irreducible) polynomials \( g(x), h(x) \) is

\[
\sum_{T=0}^{d-2} \sum_{t=0}^{T} (2|y| + 1 - |f_t|) = (2y)^{d-1} + O_f(y^{d-2}) \sim (2y)^{d-1}.
\]

We claim that almost all of these pairs of monic polynomials \( g(x), h(x) \) consist of two irreducible polynomials. Thus, \( \mathcal{R}(y) \sim (2y)^{d-1} \). We in fact establish

\[
\mathcal{R}(y) = (2y)^{d-1} + O_d(y^{d-2} \log y)
\]

by showing that there are \( O_d(y^{d-2} \log y) \) pairs of monic polynomials \( (g(x), h(x)) \) where at least one of \( g(x) \) or \( h(x) \) is reducible. Once a particular \( g(x) \) or \( h(x) \) is fixed, it determines the other. We count the ways \( g(x) \) might be reducible separately from the ways \( h(x) \) might be reducible.

First, we count the ways \( g(x) \) might be reducible. We have that \( \deg(g) = d \). Since \( h \) is monic and \( \deg h \leq d - 1 \), the equation \( f(x) = g(x) + h(x) \) implies that either \( g_{d-1} = f_{d-1} \) or \( g_{d-1} = f_{d-1} - 1 \), so in each of these cases, \( g_{d-1} \) is fixed. Since the coefficients of \( g(x) \) are bounded in absolute value by \( y \), we have met all the conditions for Lemma 1. Therefore we have that there are at most \( O_d(y^{d-2} \log y) \) monic reducible polynomials \( g(x) \).

Next, we count the ways that the monic polynomial \( h(x) \) might be reducible. Suppose we have that \( \deg(h) = d - 1 \). Since the coefficients of \( h(x) \) are bounded in absolute value by \( y \), by our remark at the end of the previous section, we have at most \( O_d(y^{d-2} \log y) \) monic reducible polynomials \( h(x) \) of degree \( d - 1 \geq 1 \). If \( \deg(h) < d - 1 \), we note that our remark also implies that the number of ways \( h(x) \) can be reducible is \( O_d(y^{d-2} \log y) \).

Thus,

\[
\mathcal{R}(y) = \left( \sum_{T=0}^{d-2} \prod_{t=0}^{T} (2|y| + 1 - |f_t|) \right) + O_d(y^{d-2} \log y) + O_d(y^{d-2} \log y) - \left( (2y)^{d-1} + O_f(y^{d-2}) \right) + O_d(y^{d-2} \log y) - (2y)^{d-1} + O_d(y^{d-2} \log y),
\]

where we have used that any constant depending only on the coefficients and degree of \( f(x) \) is small compared to \( \log y \) when \( y \) is sufficiently large. The result follows. \( \square \)
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References


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