THE $RO(\mathbb{Z}/2)$-GRADED COHOMOLOGY OF MOORE SPACES FOR CYCLIC GROUPS

CHAD GIUSTI AND WILLIAM KRONHOLM

Contents

1. Warning

Warning 1.1. This paper is a work in progress. Not all of the known results are stated here, and the known results may not be stated as cleanly as possible. The reader should keep this in mind at all times. The authors invite suggestions and corrections to the content of this paper.

2. Cohomology of Moore Spaces for Cyclic Groups

In this short paper, we investigate the cohomology of certain $\text{Rep}(\mathbb{Z}/2)$-spaces. Non-equivariantly, these are Moore spaces for cyclic groups, and so their non-equivariant cohomology is known. However, since these spaces are constructed to be $\mathbb{Z}/2$-spaces, we can also consider their $RO(\mathbb{Z}/2)$-graded cohomology, which turns out to be much more complicated. Let $H\underline{\mathbb{Z}}$ denote the $RO(\mathbb{Z}/2)$-cohomology ring of a point with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. Then a priori $H\underline{\mathbb{Z}}$ has many interesting modules. However, only certain classes of $H\underline{\mathbb{Z}}$-modules can appear as the cohomology of a $\mathbb{Z}/2$-space.

Fix M to be the Mackey functor $\underline{\mathbb{Z}}$. Suppose $B = S^{p,q}$ and we build X from B by attaching a single $(p+1, q+r)$ cell. The nonequivariant degree of the attaching map is important for determining the $H\underline{\mathbb{Z}}$-module structure of the $RO(\mathbb{Z}/2)$-graded cohomology of X.

Let us first examine the case where $r \leq 0$. As a representative case, let $r = -4$.

Denote by ω the generator of the cohomology of B and ν the generator of the cohomology of X/B. Then the E_1 page of the cellular spectral sequence for $RO(\mathbb{Z}/2)$-graded cohomology takes the form shown in Figure 1 below.

All differentials, which have bidegree (1,0), are determined by $d(\omega)$ and the $H^{*,*}(pt; \underline{\mathbb{Z}})$-module structure. Clearly, $d(\omega) = n \cdot x^2\nu$ for some integer n.

By analyzing the $H\underline{\mathbb{Z}}$-module structure, we see that we must have that

- $d(x^i\omega) = nx^{i+2}\nu$ for $i \geq 0$,
- $d(\alpha\omega/x^j) = 2nx^j\nu$ for $j = 0, 1$,
- $d(\alpha\omega/x^k) = n\alpha\nu/x^{k-2}$ for $k \geq 2$, and
- $d(\theta\omega) = d(\theta\omega/x) = 0$.

Date: December 7, 2009.

(Remember that $x\alpha = 2$.)

Depending on the parity of n, $d(y^i\omega)$ will be either 0 or $x^2y^i\nu$. In either case, nothing is yet known about $d(\theta\omega/x^j)$ for $j \geq 2$.

After taking cohomology the spectral sequence collapses and the cohomology of X is shown in Figure 2.

In this picture, the large solid dots represent either $\mathbb{Z}/n$ or $\mathbb{Z}/2n$, as appropriate. Also, the dotted lines represent piece of the module that may be nonzero, depending upon the parity of n and the behavior of the differential in the bottom cone.

Let a denote the image of ν in the cohomology of X. Then a is a generator for $\mathbb{Z}/2n$.

Consider the portion of the forgetful long exact sequence below:

$$H^{p,q-1}(X) \to H^{p+1,q}(X) \to H^{p+1}_{sing}(X) \to H^{p+1,q-1}(X)$$

According to Figure 2, this sequence is

$$0 \to \mathbb{Z}/n \to H^{p+1}_{sing}(X) \to 0$$

Thus, $H^{p+1}_{sing}(X) = \mathbb{Z}/n$. In particular, we see that n is the nonequivariant degree of the attaching map φ.

Suppose first that n is even and consider another portion of the forgetful long exact sequence:

$$0 \to H^{p,q-5}(X) \to H^{p+1,q-4}(X) \to H^{p+1}_{sing}(X) \to 0$$

Substituting the groups, this becomes

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2n \to \mathbb{Z}/n \to 0$$

If b is the generator of $H^{p,q-5}(X)$, then we can deduce that $yb = na$ and also $xyb = nxa$.

Let us examine the bottom cone further. Suppose that there is a nonzero class $c \in H^{p-1,q-7}(X)$, i.e. the differential of the E_1 page was zero on the bottom cone. Then we have a forgetful long exact sequence

$$0 \to H^{p,q-7}(X) \to H^{p+1,q-6}(X) \to H^{p+1}_{sing}(X) \to H^{p+1,q-7}(X) \to 0$$

Substituting in the groups yields the exact sequence

$$0 \to H^{p,q-7}(X) \to \mathbb{Z}/n \to \mathbb{Z}/n \to H^{p+1,q-7}(X) \to 0$$

where the middle map is the forgetful map ψ to singular cohomology. Now, the first $\mathbb{Z}/n$ is generated by αa and since $\psi(\alpha) = 2$, the middle map is multiplication by 2. Since n is even, then it must be that $H^{p,q-7}(X) = H^{p+1,q-7}(X) = \mathbb{Z}/2$.

The discussion above tells us that since n is even, all of the dotted lines do appear (as solid lines) in the cohomology. This is shown in Figure 3 below.

Observe that if $j\colon X \to X/B$ is the projection, then $j^*(\nu) = a$, and so $j^*(\alpha\nu) = \alpha a$. Thus, the $H\underline{\mathbb{Z}}$-module structure is exactly as described in Figure 3.

The general case $r \leq 0$ and even is similar to this one, the difference being the number of values of j for which $x^j a$ is infinitely divisible by y and the degree in which a' appears.

In n is odd, a similar procedure determines that $H^{*,*}(X)$ has a $H\underline{\mathbb{Z}}$-module structure as described in Figure 4.

These results are summarized below.

Proposition 2.1. *If X is formed by attaching a single $(p+1, q+r)$-cell to $S^{p,q}$, $r \leq 0$, r is even, and n is the non-equivariant degree of the attaching map of the new cell, then $H^{*,*}(X;\underline{\mathbb{Z}}) = A_{n,r}[p,q]$ where $A_{n,r}[p,q]$ in the $H\underline{\mathbb{Z}}$-module described as follows:*

If n is even, then $A_{n,r}[p,q] \cong A \oplus A'$ where

- *A is generated as a $H\underline{\mathbb{Z}}$-module by a in bidegree $(p+1, q+r)$*
- *a, xa, $\ldots$, $x^{\frac{r-2}{2}}a$ all have additive order $2n$,*
- *$x^{i+r/2}a$ has additive order n for all $i \geq 0$,*
- *$x^i y^j a$ are nonzero for all $i \geq 0$ and $j \geq 0$,*
- *na, nxa, $\ldots$, $nx^{\frac{r-2}{2}}a$, $n\alpha a/2x^i$, and $\theta a/x^i$ are all infinitely divisible by y for all $i \geq 0$,*
- *$\alpha a/x^i$ is a nonzero additive generator of additive order n for all $i \geq 0$,*
- *$\theta a/x^i y^j$, $n\alpha a/2x^i y^j$ are nonzero for all $i \geq 0$ and $j \geq 0$,*
- *yA is 2-torsion*
- *$A' \cong H\underline{\mathbb{Z}}\langle a' \rangle/(2a', \theta a', \alpha a')$ where a' has bidegree $(p+1, q+1)$.*

If n is odd, then

- *$A_{n,r}[p,q]$ is generated by a in bidegree $(p+1, q+r)$*
- *a, xa, $\ldots$, $x^{\frac{r-2}{2}}a$ all have additive order $2n$,*
- *$x^{i+r/2}a$ and $\alpha a/x^i$ have additive order n for all $i \geq 0$*
- *$x^{i+r/2}ya$ and $\theta a/x^i$ are zero for all $i \geq 0$*
- *$x^i y^j a$ is nonzero for $i = 0, \ldots, \frac{r-2}{2}$ and for all $j \geq 0$*
- *na, nxa, $\ldots$, $nx^{\frac{r-2}{2}}a$ are infinitely divisible by y.*

Of course, if $r \leq 0$ and odd, the $d(\omega) = 0$ for dimensional reasons.

Let's begin to look at the case where the attaching map is such that $d(\omega)$ is in the bottom cone of ν. Let's start with the case where $d(\omega) = n\alpha\nu/x^\ell$. The case where $\ell = 1$, is pictured in Figure 5.

Then the differential on the E_1 page of the cellular spectral sequence has the following properties:

- $d(x^{\ell+i}\omega) = 2nx^i\nu$ for all $i \geq 01$
- $d(\frac{\alpha}{x^i}\nu) = 2n\frac{\alpha}{x^{\ell+1}}\nu$ for all $i \geq 0$
- $d(x^i\omega) = n\frac{\alpha}{x^{\ell-i}}\nu$ for $0 \leq i \leq \ell - 1$
- $d(x^i y^j \omega) = 0$ for all $i \geq 0$ and $j \geq 1$.
- $d(\theta\omega) = 0$.

Taking cohomology yields the module in Figure 6.

Consider the forgetful long exact sequence:

$$H^{p,q-1}(X) \to H^{p+1,q}(X) \to H^{p+1}_{sing}(X) \to H^{p+1,q-1}(X) \to H^{p+2,q}(X)$$

Plugging in what we know this becomes the following:

$$0 \to \mathbb{Z}/n \to H^{p+1}_{sing}(X) \to \mathbb{Z}/2 \to 0$$

Given the non-equivariant cell structure of X, we know that $H^{p+1}_{sing}(X)$ must be cyclic, and so is $\mathbb{Z}/2n$. Thus, the attaching map is of even non-equivariant degree $2n$.

The module structure in this case is as described by Figure 6, with one indeterminacy. There is an extension problem to solve and in the example shown, $H^{p+1,q+1}(X)$ could be either $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4$. The general case will have many of these extension problems to solve, depending on the "q-distance" between the generators.

The final case to consider is when $d(\omega)$ hits a copy of $\mathbb{Z}/2$ in the bottom cone of ν. Explicitly, this is the case where $d(\omega) = \theta\nu/x^\ell y^k$ for some ℓ and k. This is illustrated in Figure 7 for the case $\ell = 1$ and $k = 1$. Recall that since ω has dimension (p, q), ν must have dimension $(p + k + 1, q + 2\ell + k + 3)$.

Taking cohomology yields Figure 8. In this case there is a small extension problem to solve. However, $H^{p+k+1}_{sing}(X) = \mathbb{Z}$ and so the extension of $\mathbb{Z}$ by $\mathbb{Z}/2$ must be $\mathbb{Z}$.

In Figure 8, the element a has degree $(p, q + 2\ell + 2)$ and b has degree $(p + k + 1, q + k + 1)$. Since b is not in the image of $\cdot y$, b restricts to a generator in singular cohomology. Thus $x^i b$ is nonzero for all i. In particular, $y^{k+1}a$ and $x^\ell b$ generate $H^{p+k+1,q+2\ell+k+3}(X)$. Consider the long exact sequence associated to the cofiber sequence $S^{p,q} \overset{i}{\hookrightarrow} X \overset{j}{\twoheadrightarrow} S^{p+k+1,q+2\ell+k+3}$. Since $i^*(y^{k+1}a) = i^*(x^\ell b) = y^{k+1}x^\ell\omega$, exactness implies that $j^*(\nu) = y^{k+1}a + x^\ell b$. Also j^* is a $H\underline{\mathbb{Z}}$-module map and so we have the following:

- $j^*(\frac{\theta\nu}{y^{k+1}}) = \theta a$
- $j^*(\frac{\alpha\nu}{y^{k+1}}) = \alpha a$
- $j^*(\frac{\theta\nu}{x^\ell}) = \theta b$
- $j^*(\frac{\alpha\nu}{x^\ell}) = \alpha b$

It follows that $H^{*,*}(X)$ is a free $H\underline{\mathbb{Z}}$-module with generators a and b.

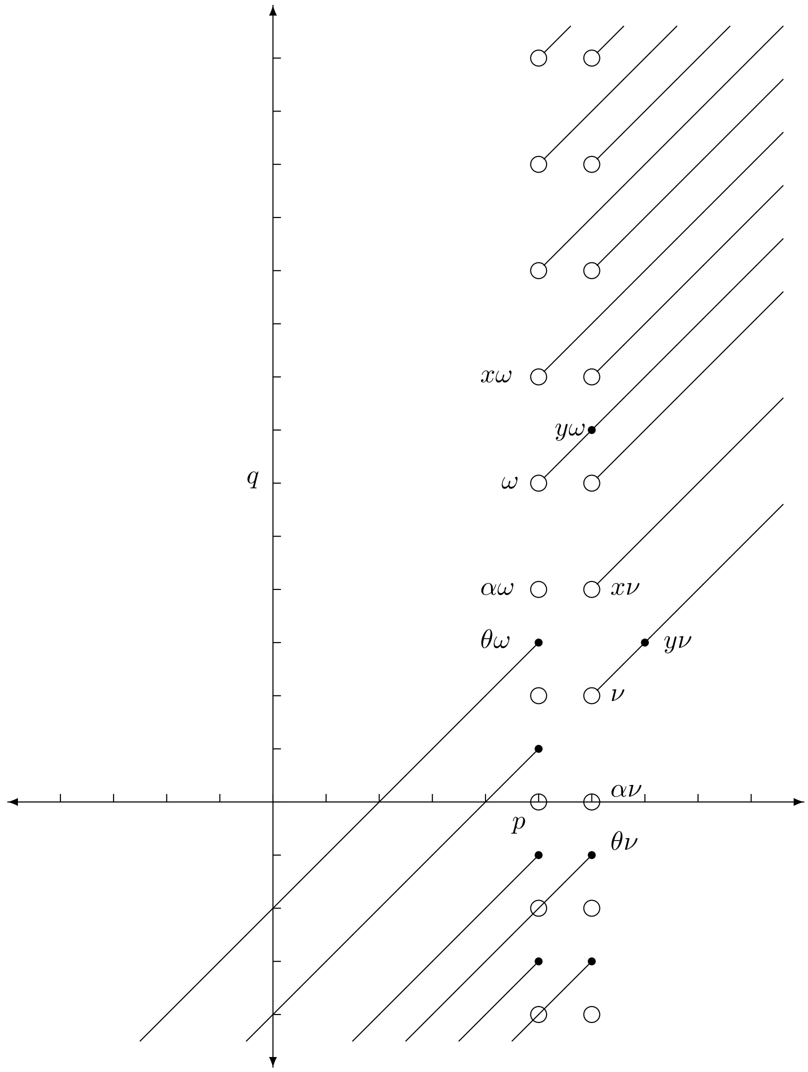

FIGURE 1. Attaching a $(p+1, q+r)$-cell to $S^{p,q}$ when $r \leq 0$.

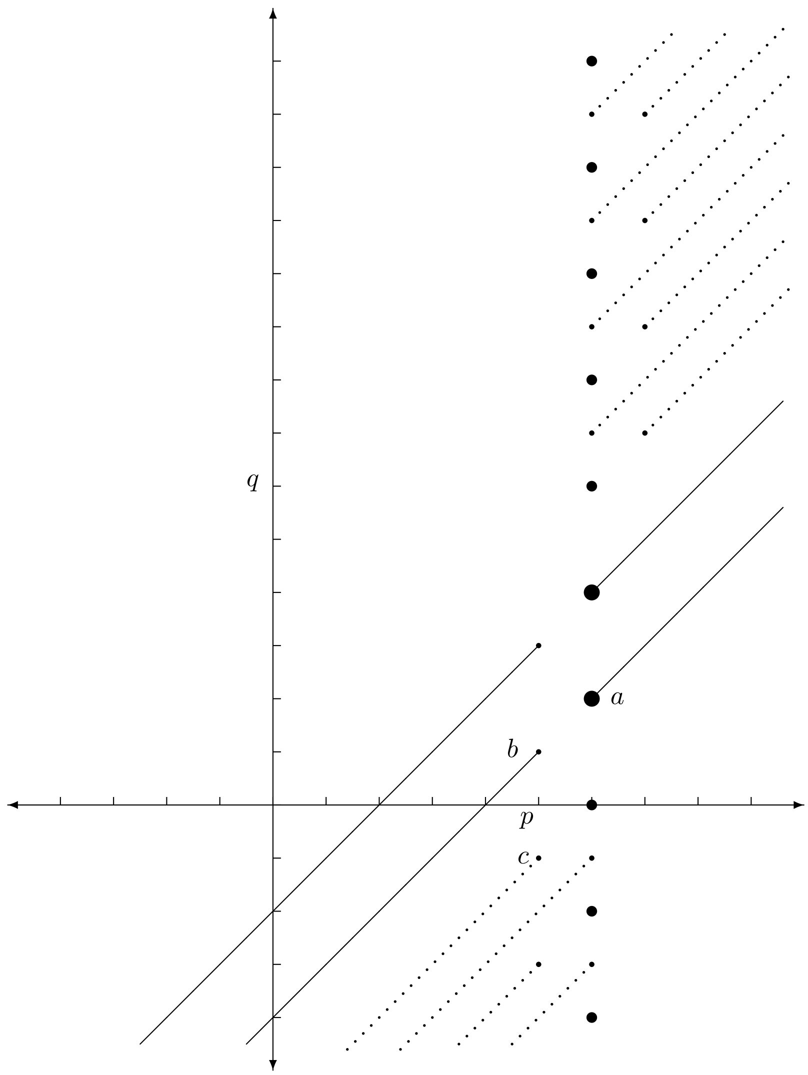

FIGURE 2. Cohomology after attaching a $(p+1, q+r)$-cell to $S^{p,q}$ when $r \leq 0$ with a degree n non-equivariant attaching map.

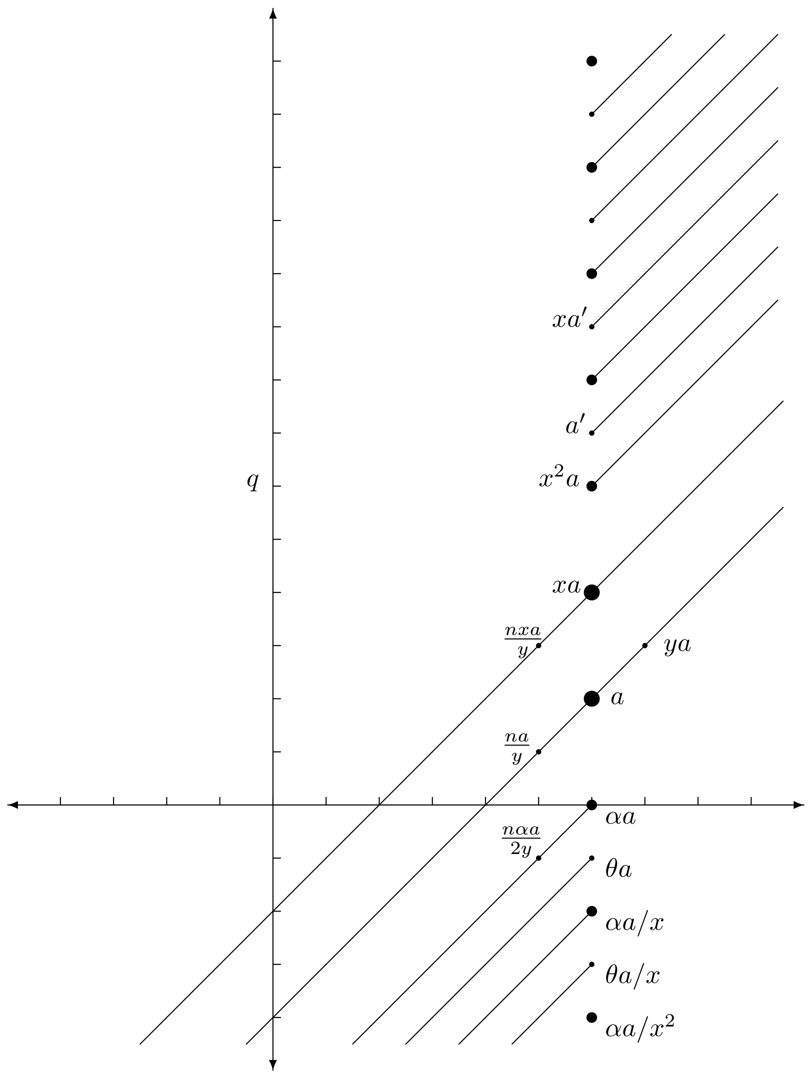

FIGURE 3. $A_{n,r}[p,q]$ (When $r \leq 0$, r is even, and the attaching map φ has nonzero even non-equivariant degree n.)

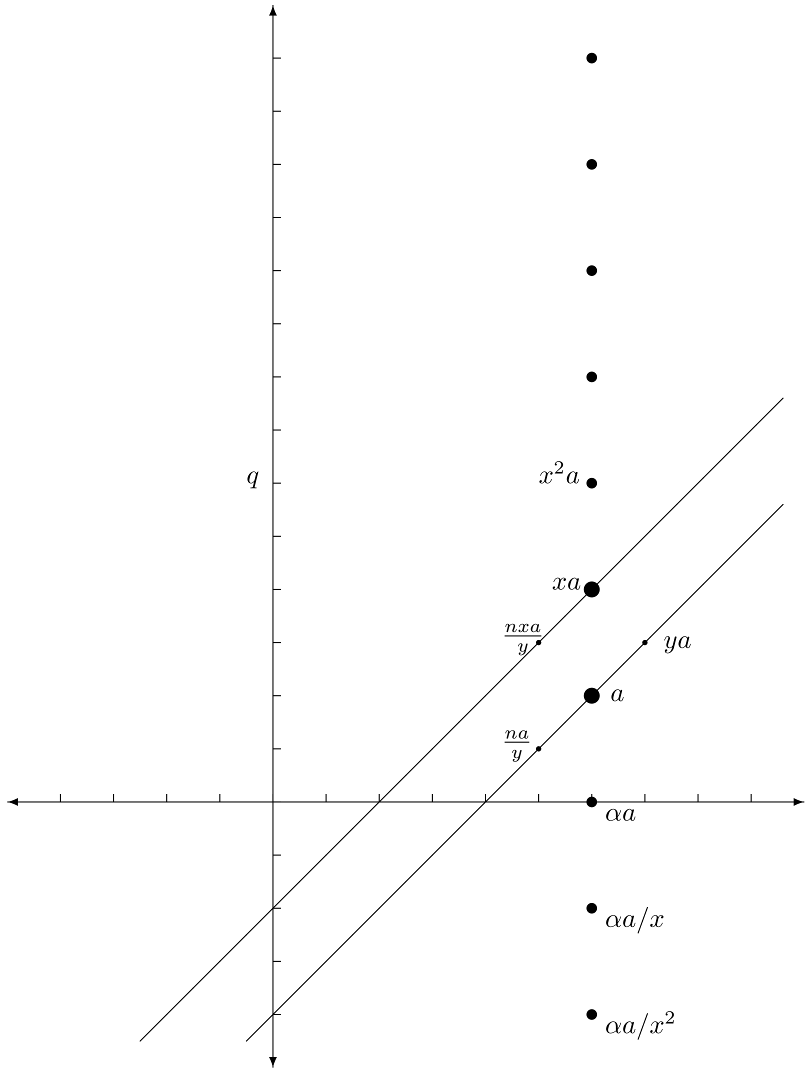

FIGURE 4. $A_{n,r}[p, q]$ (When $r \leq 0$, r is even, and the attaching map φ has odd non-equivariant degree n.)

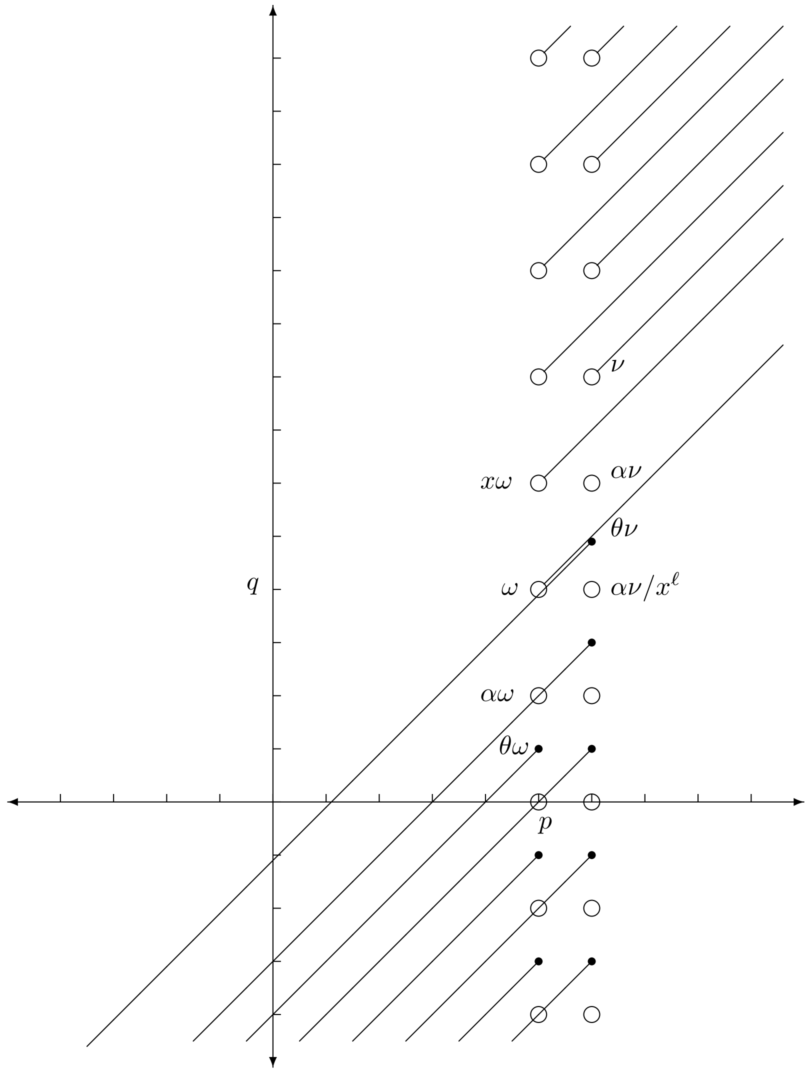

FIGURE 5. $d(\omega) = n\alpha\nu/x^\ell$, r even, $r \geq 0$, and $\ell = (r-2)/2$

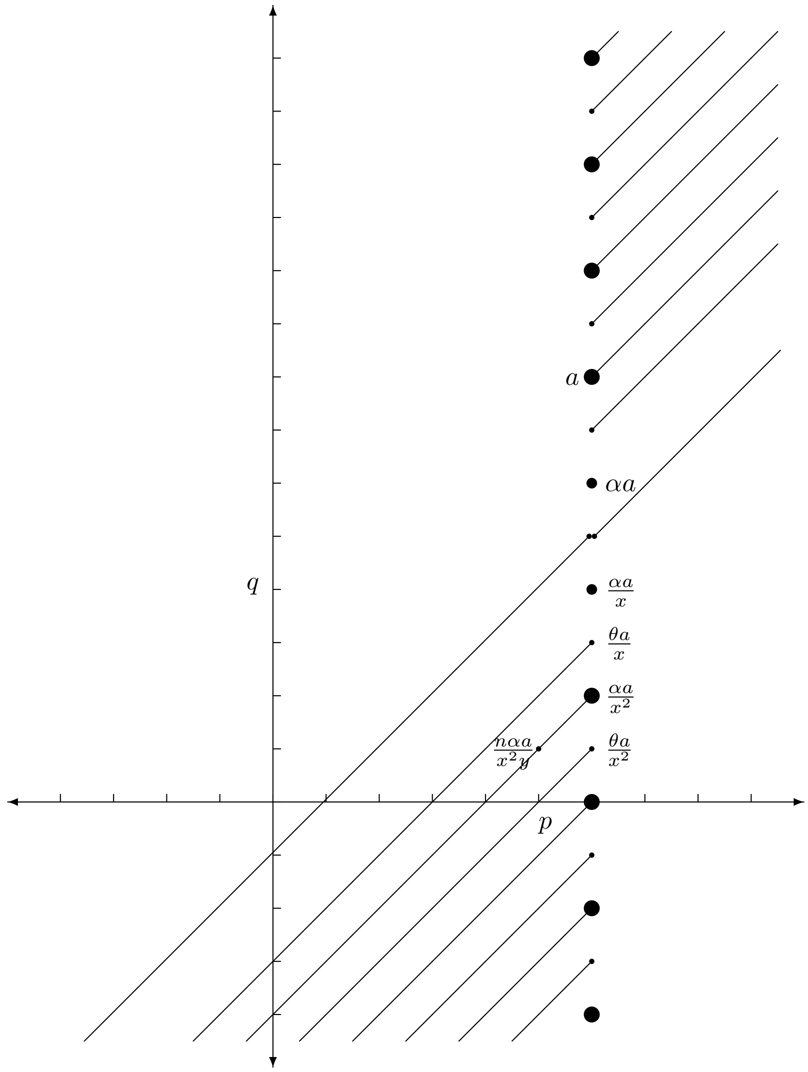

FIGURE 6. $B_{n,r}[p, q]$ (When $d(\omega) = n\alpha\nu/x^\ell$, r even, $r \geq 0$, and $\ell = (r-2)/2$)

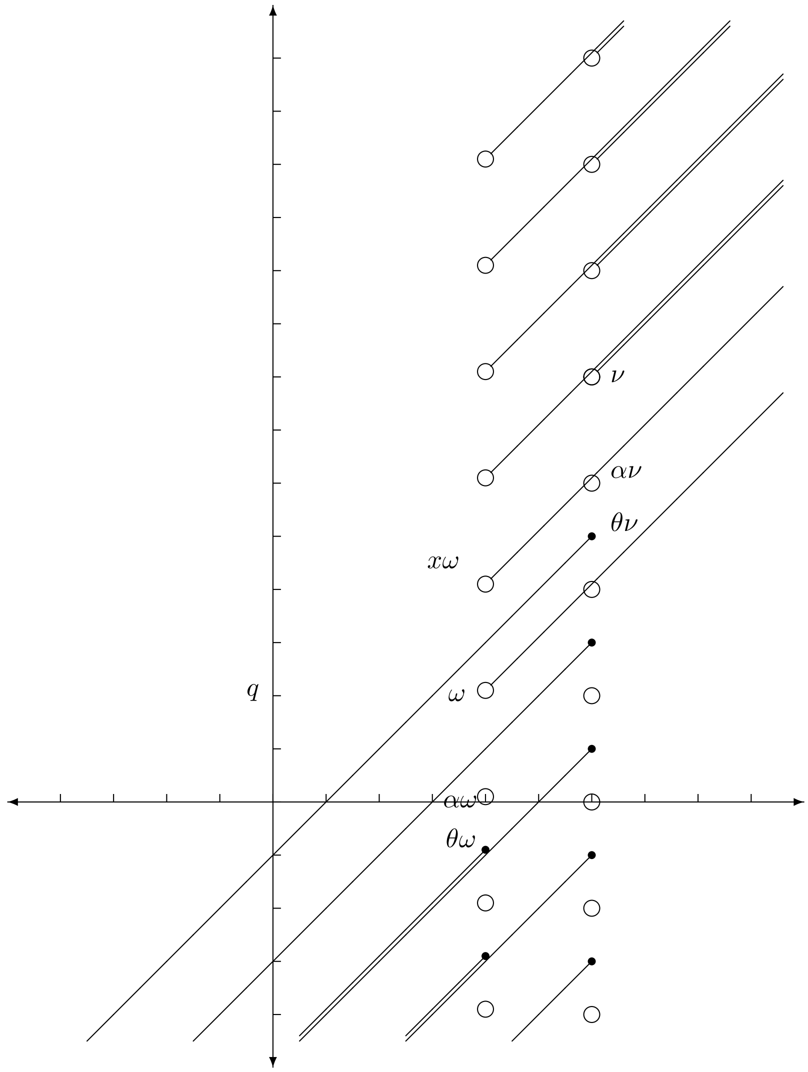

FIGURE 7. $d(\omega) = \theta\nu/x^\ell y^k$

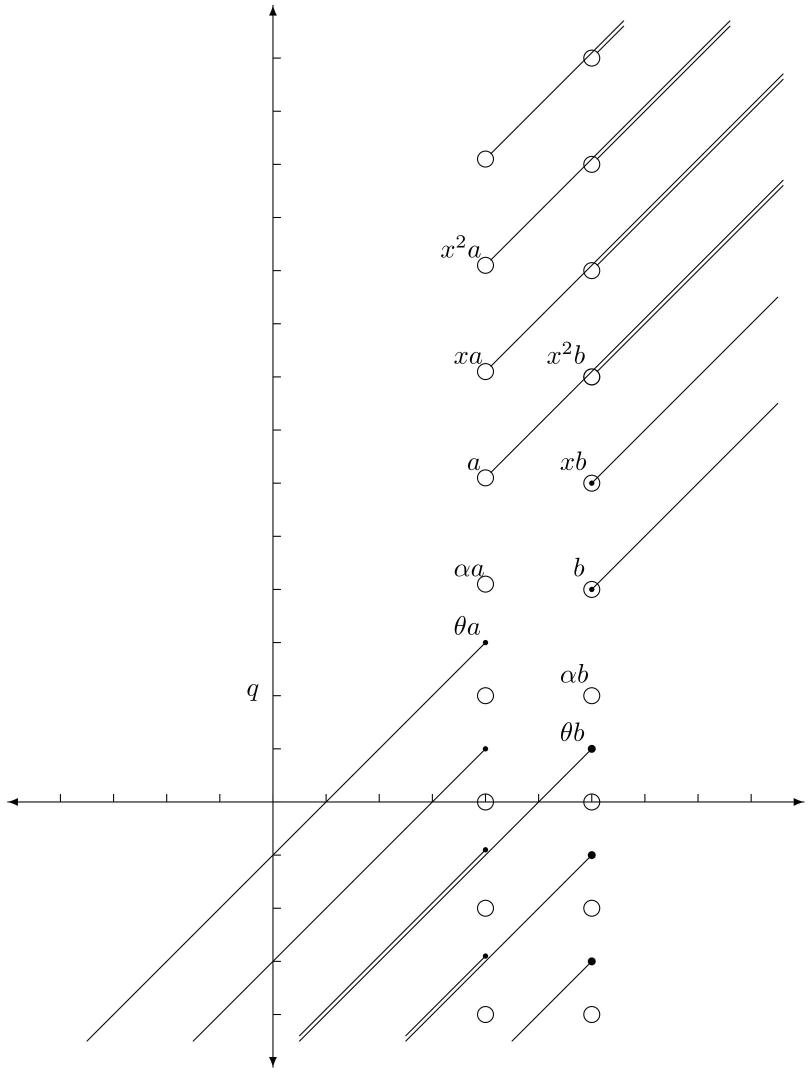

FIGURE 8. $d(\omega) = \theta\nu/x^\ell y^k$